SF1626 Calculus in Several Variables

$|\bar{x}| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ $\bar{x} \cdot \bar{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$ $\bar{x}, \bar{y} \in \mathbb{R}^3$ $\bar{x} \times \bar{y} = \begin{bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$

Surroundings $\bar{x}$: $B_r(\bar{x}) = \{\bar{a} \in \mathbb{R}^n : |\bar{x} - \bar{a}| < r\}$

Open Set: $\forall \bar{x} \in S \quad \exists B_r(\bar{x})$ for $r > 0$ "every point in set has a surronding"

Closed Set: S is closed if S' (s compliment) is open set.

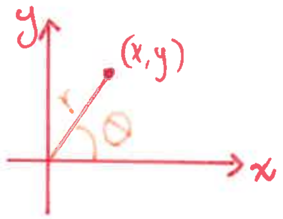
Boundary point: some $B_r(\bar{x})$ belong to S, some dont

inner point: there is at least one $B_r(\bar{x}) \subset S$

outer point: there is at least one $B_r(\bar{x}) \cap S = \emptyset$

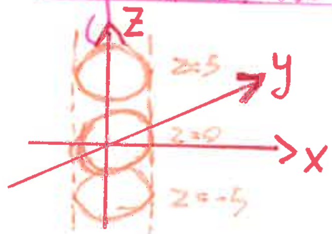
Polar Coordinates

$x = r\cos\theta \qquad y = r\sin\theta$



$r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Cylindric Coordinates



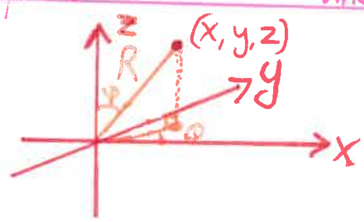
$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Spherical Coordinates



$x = R\sin\varphi\cos\theta$

$y = R\sin\varphi\sin\theta$

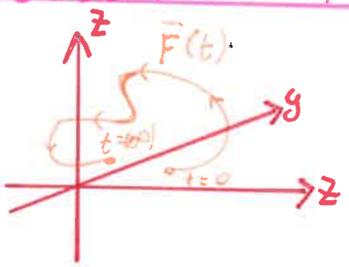
$z = R\cos\varphi$

$R = \sqrt{x^2+y^2+z^2}$

$\varphi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \quad (0, \pi)$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad (0, 2\pi)$

Vectorfunction with 1 variable



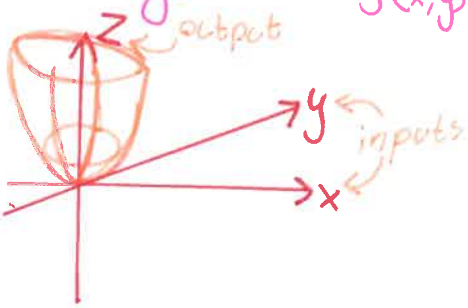
$\bar{F}(t) = (x(t), y(t), z(t))$

$\bar{F}'(t) = \lim_{h \to 0} \frac{\bar{F}(t+h) - \bar{F}(t)}{h}$

$= (x'(t), y'(t), z'(t))$

Multivariable Real Functions

$f(x,y) = z$ $\qquad f(x,y) = x^2 + y^2$

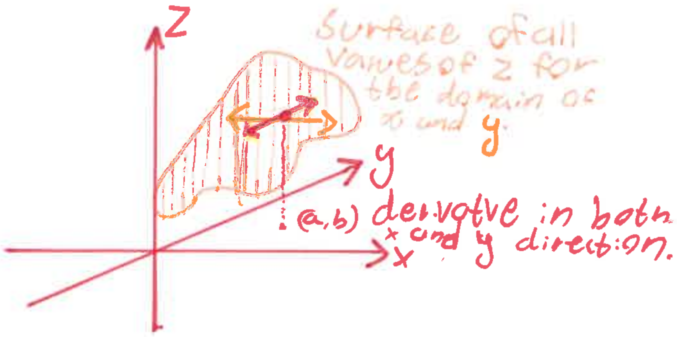


Contineous at point (a,b):

$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Partial derivatives

Derivation can be done with respect to different Variables.



$$z = f(x,y)$$

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

IF $\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$

then f is continious at (x,y)

Differentiation Rules

$u(t), v(t) \in V \quad \lambda \in \mathbb{R}$

a) $\frac{d}{dt}\left(u(t) + v(t)\right) = u'(t) + v'(t)$

b) $\frac{d}{dt}\left(\lambda(t) u(t)\right) = \lambda'(t) u(t) + \lambda(t) u'(t)$

c) $\frac{d}{dt}\left(u(t) \bullet v(t)\right) = u'(t) \bullet v(t) + u(t) \bullet v'(t)$

d) $\frac{d}{dt}\left(u(t) \times v(t)\right) = u'(t) \times v(t) + u(t) \times v'(t)$

"Product" rule

e) $\frac{d}{dt}\left(u(\lambda(t))\right) = \lambda'(t)\, u'(\lambda(t))$

f*) $\frac{d}{dt}|u(t)| = \frac{u(t) \bullet u'(t)}{|u(t)|}$

"chain" rule

* $u(t) \neq 0$

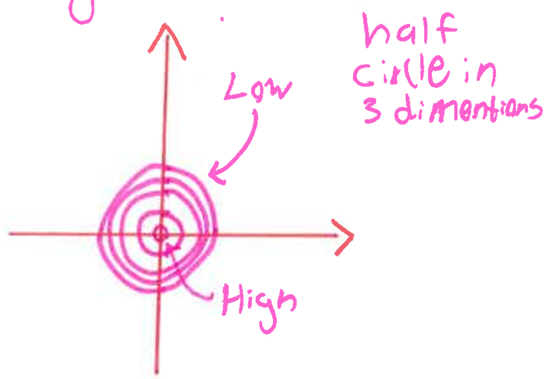
Closed/Open/Intersecting

The curve $r(t)$; $(a \le t \le b)$ is closed if $r(a) = r(b)$ else it is open.

if at any other point $r(x) = r(y)$, $x \neq y$ then the curve is self-intersecting

Level Curve

Wiev the topdown and draw lines of equal heights (contour lines)



Tangent Plane

The two partial derivatives at a point (x,y,z) of $f(x,y)$ can be used to find tangent vectors and thus a tangent plane using the vectors:

$$T_1 = \begin{bmatrix} 0 \\ 1 \\ f_2(a,b) \end{bmatrix} \quad T_2 = \begin{bmatrix} 1 \\ 0 \\ f_1(a,b) \end{bmatrix}$$

Thus the equation of the Plane is:

$$Z = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)$$

Arc length

The length of an arc is the derivative of the function.

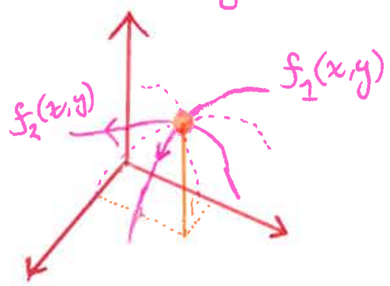
$$S = \int_a^b v(t)\,dt$$

Partial Derivatives

Given a function $f(x,y)$ the first partial derivative are:

$$f_1(x,y) = \frac{\partial}{\partial x}\left(f(x,y)\right) \quad \text{respect to } x$$

$$f_2(x,y) = \frac{\partial}{\partial y}\left(f(x,y)\right) \quad \text{respect to } y$$



Normal Vectors

If we know the two tangent lines the cross product can be used to find a Normal line to the point.

$$\vec{n} = T_1 \times T_2 = \begin{bmatrix} 0 \\ 1 \\ f_2(a,b) \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ f_1(a,b) \end{bmatrix} = f_1(a,b)\mathbf{i} + f_2(a,b)\mathbf{j} - \mathbf{k}$$

$$\vec{n} = \begin{bmatrix} f_1(a,b) \\ f_2(a,b) \\ -1 \end{bmatrix}$$

Higher Order Derivatives

Since there is more than one derivative for functions with more than one variable there is more than one derivative for each derivative. Multiple derivatives of the same variable are called Pure while other are called Mixed

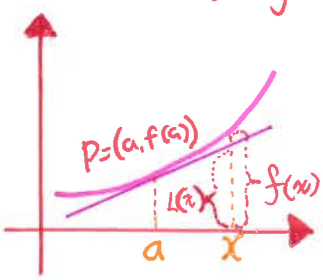
$$\left.\begin{array}{l} \frac{\partial^2}{\partial x^2} = f_{11}(x,y) \\ \frac{\partial}{\partial y^2} = f_{22}(x,y) \end{array}\right\} \text{Pure}$$

$$\left.\begin{array}{l} \frac{\partial^2}{\partial x y} = f_{21}(x,y) \\ \frac{\partial^2}{\partial y x} = f_{12}(x,y) \end{array}\right\} \text{Mixed}$$

Linear Approximation

The linearization of the function $f(x,y)$ is:

$$f(x,y) \approx L(x,y) = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(x-b)$$



Equality of Mixed Parts

The order does not matter if the derivatives are contineous around a neighbourhood of the point you are deriving.

Laplace Equation

Given a function $f(x)$ the second derivative $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y^2} = 0$ is the Laplace equation.

If f fulfils the Laplace equation it is called Harmonic.

Chain Rule

$z = f(x,y)$ and f is continious and x,y are dependent on t then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Differentiability

The function $f(x,y)$ with two variables is differentiable at the point (a,b) if:

$$\lim_{(h,k)\to(a,b)} \frac{f(a+h,b+k) - f(a,b) - h f_1(a,b) - k f_2(a,b)}{\sqrt{h^2+k^2}} = 0$$

Function from $\mathbb{R}^n$ to $\mathbb{R}^m$

given $f=(f_1, f_2, f_3 \cdots f_m)$ depending on $(f_1, f_2 \cdots f_n)$

Then f maps $\mathbb{R}_n \to \mathbb{R}_m$

The $n \times m$ derivatives $Df(x)$ can be represented in a $m \times n$ Jacobian Matrix.

$$Df(x) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_m} \end{pmatrix}$$

Multiplying with a $d\vec{x}$ can be done to find $d\vec{s}$

Implicit Functions

When we have a function of $f(x,y)$ we can express y in terms of x to get an implicit function.

$$F(x, y(x)) = f(x,y)$$

This works given that

$$f_2(a,b) \neq 0$$

since

$$\left.\frac{dy}{dx}\right|_a = -\frac{F_1(a,b)}{F_2(a,b)}$$

Composition of Functions

The chainrule can be formed with the product of Jacobian matrices.

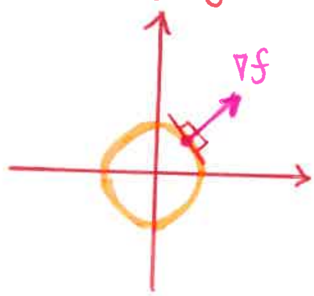
$$D(g \circ f)(x) = Dg(f(x)) \cdot Df(x)$$

This exactly mimics the single variable calculus.

Gradient Function ∇

At the point (x,y) the gradient function ∇ is defined as:

$$\nabla f(x,y) = \text{grad}\, f(x,y) = f_1(x,y)\,i + f_2(x,y)\,j$$



∇f is normal to the tangent at the point. and the direction represents the direction of the max rate of change of f, also defined by $|\nabla f|$

It is also true that if $|\bar{u}|=1$ then

$$D_{\bar{u}} f(x,y) = \bar{u} \cdot \nabla f(x,y)$$

Jacobian Determinant

Given the two functions $u=u(x,y)$ $v=v(x,y)$ the Jacobian determinant is:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Implicit function Theorem

① Given n equations with $n+m$ variables

$$n \text{ equations} \left\{ \begin{array}{l} x_1 + x_2 \cdots x_n \cdots x_{n+m} = 0 \\ x_1 + x_2 \cdots\cdots = 0 \\ \vdots \\ x_1 + x_2 \cdots\cdots = 0 \end{array} \right\} \begin{array}{l} F_1 \\ \vdots \\ F_n \end{array}$$

② And a point $p=(a_1, \cdots a_n, \cdots a_{n+m})$

③ And all functions have contineous partial derivatives with respect to each variable near point p

④ Finally suppose:

$$\left.\frac{\partial(F_1 \cdots F_n)}{\partial(x_{n+1}, x_{n+2} \cdots x_{n+m})}\right|_{p_0} \neq 0$$

THEN:

The system can be solved for $(x_{n+1}, x_{n+2}, x_{n+3} \cdots x_{n+m})$ as a function of $(x_1 \cdots x_n)$ such that:

$\phi_j(x_1, x_2, x_3 \cdots x_n) = a_{n+j}$ ϕ_j being one o these func

Moreover:

$$\frac{\partial \phi_i}{\partial x_j} = \frac{\partial x_{n+i}}{\partial x_j} = \frac{-\frac{\partial(F_1 \cdots F_n)}{\partial(x_{n+1} \cdots x_{n+m})}}{\frac{\partial(F_1 \cdots F_n)}{\partial(x_{n+1} \cdots x_{n+m})}}$$

Extreme Values

a function $f(x,y)$ can have an extreme value at point (a,b) iff one of the following applies.

a) a critical point satisfying $\nabla f(a,b)=0$

b) a singular point where $\nabla f(a,b)$ is nonexistent

c) a boundry point of the domain of f.

If f is contineous and its domain is both closed and unbounded then there ~~must~~ must be an extreme value in f where $f(x,y)$ takes on a max, and min value.

Second Derivative Test

Suppose $\bar{a}=(a_1, a_2 \dots a_n)$ is a critical point of $f(x_1, x_2 \dots x_n)$ and f is contineous in a neighbourhood arround $\bar{a}$. Then the Hessian Matrix is:

$$\mathcal{H}(\bar{x}) = \begin{pmatrix} f_{11}(\bar{x}) & f_{12}(\bar{x}) & \dots & f_{1n}(\bar{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(\bar{x}) & f_{n2}(\bar{x}) & \dots & f_{nn}(\bar{x}) \end{pmatrix}$$

Since $f_{12}=f_{21}$ because of the continuity the matrix is symmetrical.

1) $\mathcal{H}(a)$ is positve definate $\Rightarrow f$ has a local minimum at a

2) $\mathcal{H}(a)$ is negative definite $\Rightarrow f$ has a local maximum at a

3) $\mathcal{H}(a)$ is indefinate a is a saddle point

4) $\mathcal{H}(a)$ is not 1), 2) or 3) no information is given by this test.

Taylor Formula.

like in singlevariable calculus taylors-equation can be used to approximate a surface in three dimension but also in higher dimensions around a point (a,b). With decreasin accuracy as you tread away from the point (a,b). The resulting formula is always a polynomial.

$$f(a,b) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial f}{\partial y}\frac{\partial f}{\partial x}(a,b)(x-a)(x-b) + \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2\right)$$

The error of the formula is proportional to:

$$\text{error} \sim O\left(\sqrt{(x-a)^2+(y-b)^2}^{\,3}\right)$$

Critical Points

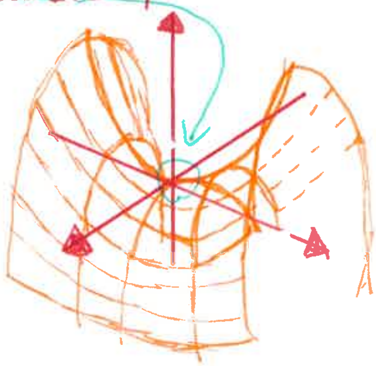
A critical point can either be a local minimum, maximum or a saddle point. To find which type the Δf can be used.

$$\Delta f = f(a+h, b+k) - f(a,b)$$

$\Delta f \leqslant 0$ $\forall$ small $h,k \Rightarrow$ Maximum

$\Delta f \geqslant 0$ $\forall$ small $h,k \Rightarrow$ Minimum

if Δf is both positive and negative it is a saddle point



Positiv/Negativ/indefinate matrix

If all eigenvalues of the matrix is positive/negative the matrix is positive/negative definate.

If there is a mix of positive and negative it is indefenate

If there is any amount of 0 the matrix is none of the above.

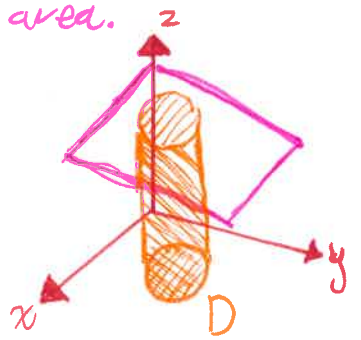
Extreme Values w/t restrictions

given a function f with a restricted domain. Extreme values are found by:

1) find critical/singular points inside of the domain of f
2) Parametrize the boundary of the domain (into pieces if necessary) and express f as a function of this.
3) evaluate the points found in 1 & 2.

Doubble Intergrals

Instead of finding area under a curve, double intergrals find the volume under an area.



$$\iint_D f(x,y)\,dx\,dy$$

f is continuous on a closed bounded domain D.

Integration over simples

given f(x,y) is contineous

① in a y simple domain D: given by $a \leq x \leq b$, $c(x) \leq y \leq d(x)$

$$\iint_D f(x,y) = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$

Lagrange Multipliers

When the range is limited rather than the domain the lagrange function is used. It works by finding the point on the curve with a derivative tangent to the restriction.

$$\mathcal{L}(x,y,\lambda) = \underbrace{f(x,y)}_{\text{function to maximize.}} + \overbrace{\lambda}^{\text{lagrange multiplier}} \underbrace{g(x,y)}_{\text{function of the restricted range}}$$

When all derivatives are equal to zero, (x,y) are maximized in f(x,y) restricted on g(x,y)=0

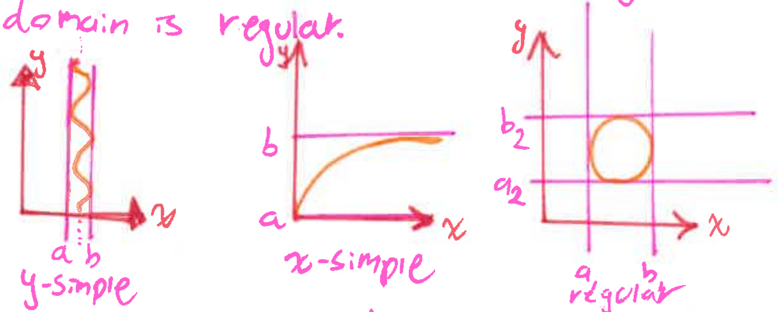
$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

This can be extended to more constraints by introducing more variables and functions to $\mathcal{L}$

x/y simple

a domain D in $\mathbb{R}^2$ is y-simple if all values can be restricted between $x=a$ and $x=b$. Vice versa for x-simple.

If a domain is both x- and y-simple the domain is regular.



all contineous functions are intergrable over a regular domain.

② in an x-simple domain D given by $c \leq y \leq d$ and $a(y) \leq x \leq b(y)$

$$\iint_D f(x,y) = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy$$

Mean Value Theorem

$f(x,y)$ is contineous on the bounded closed connected set D then a point (x_0, y_0) such that:

$$\iint_D f(x,y)\,dA = \underbrace{f(x_0, x_y)}_{\text{mean value of } f \text{ over } D} \times D\ \text{area}$$

$$\bar{f} = \frac{1}{\text{Area of D}} \iint_D f(x,y)\,dD$$

Variable Change

Polar Coordinates is just one very useful example of a base change of variables, but it can be done on any function that bijectively maps onto x and y.

$$x = x(u,v) \qquad y = y(u,v)$$

Thus x and y also can map onto u,v

$$u = u(x,y) \qquad v = v(x,y)$$

This method however must also account for the new rate of change of u,v compared to $dy\,dx$. This happens to be the Determinant of the Jacobian matrix of dA.

$$dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \cdot du\,dv$$

This means, in general:

$$\iint_D f(x,y)\,dx\,dy = \iint_S g(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$f(u(x), v(y)) = g(u,v)$$

Polar Coordinates

In many scenarios using polar coordinates is easier than rectangular coordinates. Then x, y must be substituted.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$x^2 + y^2 = r^2 \qquad \frac{y}{x} = \tan\theta$$

$$dx\,dy = dA = r\,dr\,d\theta$$

This strategy is used to solve the following.

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Triple Intergrals

extending calculus to three variables from two is quite straightforward. Instead of finding a volume, a 4d hypervolume is found.

Variable change (3 intergrals)

Similar to change in 2 dimentions a function $x = x(u,v,w)$, $y(u,v,w)$, $z = z(u,v,w)$ needs to be formed. Similarly the new rate of change needs to be acounted for.

$$dV = dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

In general

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_S g(u,v,w) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Spherical coordinates

in a tripple intergral a variable change to spherical coordinates is very useful.

$$x = R\sin\phi\cos\theta \qquad dV = dx\,dy\,dz = R^2\sin\phi\,dR\,d\phi\,d\theta$$

$$y = R\sin\phi\sin\theta \qquad R = \text{radius} \in \mathbb{R}$$

$$z = R\cos\phi \qquad \phi = z\ \text{axis angle} \in [0, \pi]$$

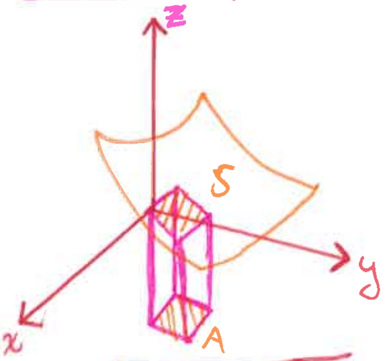
$$\theta = \text{polar angle} \in [0, 2\pi]$$

Derives to:

$$x^2 + y^2 + z^2 = R^2$$

$$r = \sqrt{x^2 + y^2} = R\sin\phi$$

Surface Area in 3D



$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

=>

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Vector Fields

A function whose domain and range are subsets of $\mathbb{R}^3$ is called a vector field. A Vector is assigned to each Point in the field. This is done with the function F.

$$F(x,y,z) = F_{1*}(x,y,z)\,\hat{i} + F_{2*}(x,y,z)\,\hat{j} + F_{3*}(x,y,z)\,\hat{k}$$

* components, not partials

They are considered smooth if each component has a continious partial derivative.

Field Lines

given a field in any $\mathbb{R}^n$ field lines are parallell to the field F but lacs magnitude

Conservative Fields

Every gradient of a scalar field is a vectorfield but not every vectorfield has a scalar field.

$$F(x,y,z) = \nabla \phi(x,y,z) = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

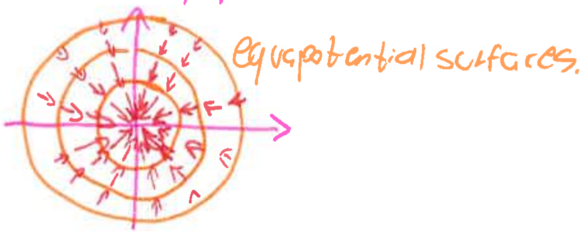
If this is true, F is a conservative field in a domain D

It is true for all conservative fields that:

$$\frac{\partial}{\partial y} F_1(x,y) = \frac{\partial}{\partial x} F_2(x,y)$$

Equipotential Surface

Surfaces of equal potental for a field are called equipotental surfaces.



Sources, Sinks & Dipoles

Sources add potential to a vector field, sinks remove the potential. Dipoles are systems of both sinks and sources.

Line Intergrals

describes the length of an arc evaluated by:

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| dt$$

for vectorfields this extends to

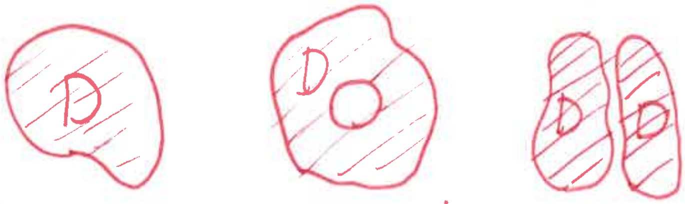
$$\int_C \vec{F} \cdot d\vec{r} = \int_C F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz$$

If the curve is closed it is often indicated by

$$\oint \vec{F} \cdot d\vec{r}$$

Connected & simply connected Domain

Connected => every point in D is connected to every other point.
simply connected => connected and no holes.



simply connected | connected | not connected

Independence of Path

Let D be an open connected Domain
Let F be a smooth vectorfield defined on D. Then it is equivalent:

1) F is conservative in D
2) $\oint_C F \cdot dr = 0$ for every piecewise smooth closed curve C in D
3) between any two points in D $\int_C F \cdot d\vec{r}$ has the same value for every piecewise smooth curve.

Line integral conservative field

because of the definition on the left. the integral of a line in a conservative field is:

$$\int_C F \cdot dr = \int_C d\phi = \phi(P_1) - \phi(P_2)$$

Line Integral in Vector field

This is essentially doing the dot product of a line in a field with the field vectors. This time the vector of the input wrt respect to $\hat{i}\,\hat{j}\,\hat{k}$ are dotted with a line.

$$F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$$

$$F(t) = \begin{bmatrix} r_x(t) \\ r_y(t) \\ r_z(t) \end{bmatrix}$$ The path taken through F

$$\hat{r}(t) = \begin{bmatrix} r_x'(t) \\ r_y'(t) \\ r_z'(t) \end{bmatrix} \cdot \frac{1}{dt} = \frac{d\bar{r}}{dt}$$ The direction vector of $\bar{r}$ at any time t.

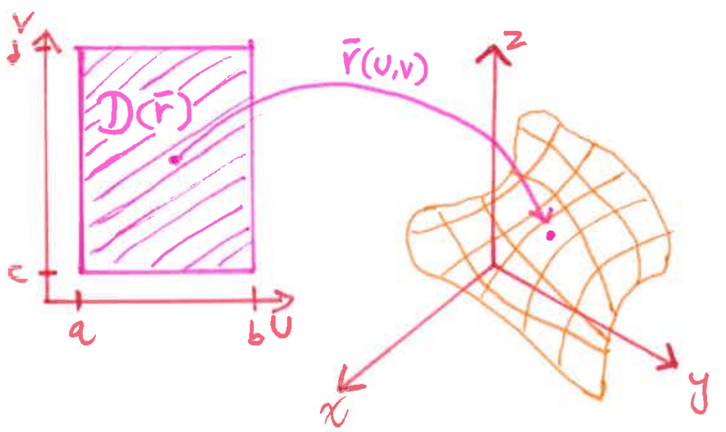
$$\int_{C \text{ - curve } r} F \cdot d\bar{r} = \int_{t=a}^{t=b} F \cdot \frac{d\bar{r}}{dt}\, dt$$

$$= \int_{t=a}^{t=b} \Big(P(r_x(t), r_y(t), r_z(t)) \cdot \frac{dx}{dt} + Q(r_x(t), r_y(t), r_z(t)) \cdot \frac{dy}{dt} + R(r_x(t), r_y(t), r_z(t)) \frac{dz}{dt} \Big)\, dt$$

Parametric Surfaces

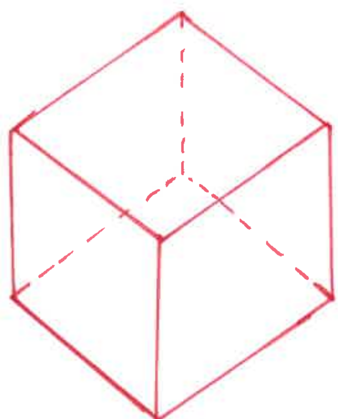
in 3D-Space a parametric surface is a function $\bar{F}$ defined on a rectangle given by $a \le u \le b$ and $c \le v \le d$ in the uv plane as the values in $\mathbb{R}^3$

$$F(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$



Composite surfaces

Two or more surfaces can be joined to form a composite surface. The surfaces must be joined together at their boundaries. Closed surfaces have no edges, I.e. cannot be joined



Composit Shape of 6 sides

Surface Intergrals

The surface intergral finds the Area of a parametric surface. For now this involves the sum of planes section that can approximate the surface.

Let S_i be a part of the surface S. Then the Riemann sum of the function $f(x,y,z)$ is:

$$R_n = \sum_{i=1}^{n} f(x_i, y_i, z_i) \cdot \Delta S_i$$

where ΔS_i is the area of S_i

Normal vector to Smooth S

Since any point on the surface has two tangent vectors in the form $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$, their cross product must be normal to the surface.

$$\vec{n} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$$

$$= \frac{\partial(y,z)}{\partial(u,v)}\hat{i} + \frac{\partial(z,x)}{\partial(u,v)}\hat{j} + \frac{\partial(x,y)}{\partial(u,v)}\hat{k}$$

Also the so called area element is then given by:

$$dS = |\vec{n}|\, du\, dv$$

Thus the area of S is the infinite sum of dS, IE the integral.

$$\text{Area of } S = \iint_S dS$$

Area over a function is thus.

$$\iint_S f(x,y,z) \cdot dS = \iint_D f(\vec{r}(u,v))\,|\vec{n}|\, du\, dv$$

$$= \iint_D f(x(u,v), y(u,v), z(u,v)) \cdot \sqrt{\frac{\partial(y,z)}{\partial(u,v)}^2 + \frac{\partial(z,x)}{\partial(u,v)}^2 + \frac{\partial(x,y)}{\partial(u,v)}^2}\, du\, dv$$

Smooth Surfaces

A set S in $\mathbb{R}^3$ is a smooth surface if any point p in S has a neighbourhood (N) that is the domain of a function $g(x,y,z)$ satisfying.

1) $N \cap S = \{\vec{Q} \in N \mid g(\vec{Q}) = 0\}$
2) $\nabla g(\vec{Q}) \neq \vec{0}$ if $Q \in N \cap S$

This must be true for any interior point.

For surfaces that cannot be parametized easily. The area element dS can be found

$$dS = \left|\frac{1}{\cos\gamma}\right| dx\,dy = \frac{|\vec{n}|}{|\vec{n} \cdot \hat{k}|}\, dx\,dy \qquad k \text{ is } = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Where γ is the angle between $\vec{n}$ and the positive z axis.

If a surface S with equation $G(x,y,z) = 0$ The normal vector $\vec{n}$ can be found by:

$$\vec{n} = \nabla G(x,y,z)$$

Since $\vec{n} \cdot \hat{k} = G_3$ then

$$dS = \left|\frac{\nabla G}{G_3}\right| = \left|\frac{\vec{n}}{\vec{n} \cdot \hat{k}}\right| dx\,dy$$

General Case for Spheres

The area element of a sphere can be generalized by:

sphere of radius $R = a$

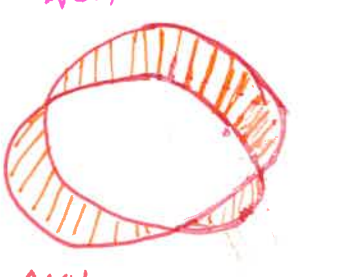
$$dS = a^2 \sin\phi \; d\phi\, d\theta$$

Oriented Surfaces

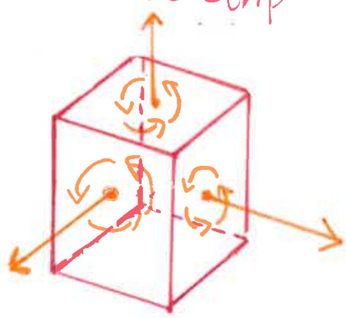
A smooth surface $\mathcal{S}$ is said to be orientable if there exists a unit vector field $\hat{N}(P)$ such that any point P on $\mathcal{S}$ the value of $\hat{N}(P)$ is normal to $\mathcal{S}$. $\hat{N}(P)$ must be contineous. The side $\hat{N}$ points to is the positive side. An oriented surface is the combination of the field and smooth surface.

Not every surface composition can be oriented. The rotation with the righthand rule must be oposite at the edges

Non orientable



Möbious Strip



orientable

Finding Flux of Vectorfield across Surface

The flux is the sum of all vectors flowing into/out of a surface from a vectorfield dotted with the normal to the field.

$$\text{Flux} = \iint_{\mathcal{S}} \vec{F}(x,y,z) \cdot \hat{N}(x,y,z)\, dS$$

If the equation for the surface is implicit wit reference to another variable. $z = f(x,y)$ then:

$$\hat{N} = \pm \frac{-\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \hat{k}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

Thus:

$$\partial \vec{\mathcal{S}} = \hat{N}\, dS = \pm\left(-\frac{\partial f}{\partial x}i - \frac{\partial f}{\partial y}j + k\right) dx\, dy$$

Curl

Curl of a vectorfield is also a vector. it is proportional to the rotation of the vectors in the field.

$$\text{Curl } F = \nabla \times F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k$$

Gradience

First order information of rate of change, of a scalar field.

$$\text{grad } f(x,y,z) = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Divergence

Since a field of vectors has more than one gradient for each component of their vectors More complex ways of analysis is needed.

$$\text{div } F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

in this case F_1, F_2, F_3 are each component of F and the divergence is a scalar.

In 2D

$$\text{div } F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

$$\text{curl } F = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k$$

Interpretation Of Div

Divergence can be thought of as the tota flux going away from a single point or an infinitly small circle around a point.

$$\operatorname{div} F(P) = \lim_{\varepsilon \to 0^+} \frac{3}{4\pi \varepsilon^3} \oiint F \bullet \hat{N}\, ds$$

where $\hat{N}$ is normal to the sphere.

Identities

let ϕ, ψ be two scararfields
let $\bar{F}, \bar{G}$ be two vectorfields
Smoot with partial derivatives.

a) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$ Product rule

b) $\nabla \bullet (\phi \bar{F}) = (\nabla\phi)\bullet\bar{F} + \phi(\nabla\bullet F)$

c) $\nabla \times (\phi F) = (\nabla\phi)\times\bar{F} + \phi(\nabla\times\bar{F})$

d) $\nabla\bullet(F\times G) = (\nabla\times F)\bullet G - F\bullet(\nabla\times G)$

e) $\nabla\times(F\times G) = (\nabla\bullet G)F + (G\bullet\nabla)F - (\nabla\bullet F)G - (F\bullet\nabla)G$

f) $\nabla(F\bullet G) = F\times(\nabla\times G) + G\times(\nabla\times F) + (F\bullet\nabla)G + (G\bullet\nabla)F$

g) $\nabla\bullet(\nabla\times F) = 0$

h) $\nabla\bullet(\nabla\phi) = \hat{0}$

i) $\nabla\times(\nabla\times F) = \nabla(\nabla\bullet F) - \nabla^2 F$

Divergene in A Plane

let R be a region in xy-plane.
let $\hat{N}$ be the unit normal of the boundary $\mathcal{C}$. $F = F_1(x,y) + F_2(x,y)$

$$\iint_R \operatorname{div} F\, dA = \oint_{\mathcal{C}} F \bullet \hat{N}\, dr$$

Interpretation Of Curl

Curl measures the extent the vector field "swirls" around point P.
let $\mathcal{C}_\varepsilon$ be a circle of radius ε and normal $\hat{N}$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\pi\varepsilon^2} \oint F\bullet dr = \hat{N}\bullet \operatorname{Curl} F(P)$$

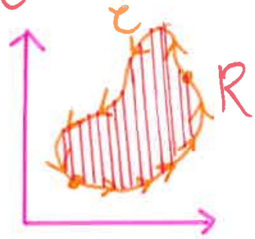
Solenoidal and irrotational fields

A vectorfield is solenoidal in a domain D if $\operatorname{div} F = 0$ in D.
A vectorfield is irrotational in a domain D if curl $F = \bar{0}$ in D
Every conservative vectorfield is irrotational
The curl of any vectorfield is solenodal.

Greens Theorem

Let R be a regular closed region in the xy plane whose boundary going anticlockwise is $\mathcal{C}$. $F = F_1(x,y)\hat{i} + F_2(x,y)\hat{j}$

$$\oint_{\mathcal{C}} F_1(x,y)\,dx + F_2(x,y)\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$



Thus:

$$\oint_{\mathcal{C}} x\,dy = -\oint_{\mathcal{C}} y\,dx = \frac{1}{2}\oint_{\mathcal{C}} x\,dy - y\,dx = \iint_R 1\, dA = \text{area of } R$$

Divergence Theorem 3D

Let D be a domain with boundry S with unit vector $\hat{N}$ pointing outward. And F is a vectorfield.

$$\iiint_D \operatorname{div} F \, dV = \oiint_S F \bullet \hat{N} \, ds$$

Variants of divergenc Theorem

If the conditions on the left apply and ϕ is a scalar field then:

a) $$\iiint_D \operatorname{curl} F \, dV = -\oiint_S F \times \hat{N} \, ds$$

b) $$\iiint_D \operatorname{grad} \phi \, dV = \oiint_S \phi \hat{N} \, ds$$

Stokestheorem

Let S be a surface in 3D space with unit normal $\hat{N}$ and boundary C and F is a vectorfield. then

$$\oint_C F \bullet dr = \iint_S \operatorname{curl} \vec{F} \bullet \hat{N} \, ds$$