Problem Description

A teacher has 5 different sets of homework in list of students N students and has another list P different pairs of students who have worked together in previous courses. The task is to assign a different homework to each student so that the number of pairs $|P|$ where both students are assigned the different homework is maximised.

Problem Reduction

The problem can be modelled using graph theory, create a vertex for each student and for each pair of students create an edge between their corresponding vertex. The 5 different homework sets can be thought of as 5 different colours and the goal is to colours the graph without directly adjacent vertices having the same colour. Note that this graph is not necessary a planar graph and the 4 colour theorem does not apply.

0.8-Approximation

Given an optimal algorithm OPT for the problem, this algorithm would not necessarily assign each student a homework such that there are no former peers who gain the same homework, but rather it would maximize the number of peers who do not get the same homework. To create a 0.8 approximation of this the new algorithm ALG would need to avoid a collisions at least 0.8 of the times when OPT would do. This form of 0.8 approximation is done in the following algorithm.

Algorithm 1 0.8 Homework Assignments

```
Input: N, H
for n ∈ N do // Go through the list of students
    h ← rand(H) // pick a random homework from the set of homework problems
    assign h to n // Assign the random homework to the current student
end for
```

What this very short algorithm which takes the list of students N as the first argument and then the list of homework problems H as a second argument. It proceeds by taking each student n from the list of students and assigns them a random homework h from the list of homework sets H.

Proof of Correctness

Observation

Given an optimal solution OPT it does not actually matter what homework that is assigned to each student but rather the relation between the assignments, in this regard if all students who were assigned homework h_i switched with students with homework h_j the solution would still be optimal. This means its actually the of assignments or two students who have worked together or edges in the graph between two students that matters instead of each individual assignment for each student. Additionally let P be the number of pairs of students who have worked together in the past, OPT can at most satisfy all of them and which means $P \geq OPT$

Correctness

Let OPT be the assignments that yield the most pairs of students that have not worked together in the past. After the algorithm has terminated and all students have a random homework problem assigned to them. At this point the expected value of ALG is the fraction of pairs where both students were not assigned the same homework. Given a pair p of two students assigned with the homework h_i and h_j the the expectation that p will be satisfied $E[h_i = h_j] = \frac{1}{5}$ which means that $E[h_i \neq h_j] = 1 - E[h_i = h_j] = \frac{4}{5} = 0.8$. Since the entire algorithm uses the sum of these expectations, the *linearity of expectations* can be used to find that the expected value for the number satisfied pairs

$$E[ALG] = \sum_{h_i, h_j \in p}^{P} E[h_i \neq h_j] \overset{loe}{=} E[h_i \neq h_j] \times P = 0.8P$$

where P is the number of pairs of students. From the observation it follows that $P \geq OPT$ which means $E[ALG] = 0.8P \geq 0.8OPT$ or in short, $E[ALG] \geq 0.8OPT$

Complexity

Since there *for* loop is iterated once for each students and a random homework can be selected in constant time the runtime of the complexity is $\mathcal{O}(|N|)$ where $|N|$ is the number of students.